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Derivation of Partial Amplitudes and the Validity of Dispersion Relations for Production Processes*

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Certain desirable requirements lead to an essentially unique definition for partial-wave amplitudes in production processes. This definition is given in concrete form and the case of particles with spin is treated. An application is included as an example of the usefulness of the partial amplitudes. A dispersion relation is given that is satisfied by the complete amplitude and probably also by the partial amplitudes.

1. INTRODUCTION

INVESTIGATIONS of scattering amplitudes (corresponding to two particles coming in and two emerging) from the S matrix standpoint have reached an encouraging stage. The formulation of the problem of their determination now seems to be almost understood, in terms of basic properties of analyticity, crossing symmetry, and unitarity.[1] Also, its solution gives promise of being tractable, largely because it seems possible to postulate so much analyticity for the matrix elements. The problem of production amplitudes (where more than two particles emerge), with which one is concerned in its own right and also to complete the scattering problem, is in a much less happy state. This is partly because of the presence of so many more variables and partly because of the complicated analyticity properties of the amplitudes,[2] which are largely the result of final-state interactions.

In the case of scattering, one prefers to work not with the complete amplitude but with the partial-wave amplitudes. Among the desirable properties of these are:

A. They seem to have an easy physical interpretation.

B. The series expansion of the complete amplitude in terms of the partial amplitudes converges throughout the physical region of the cosine of the scattering angle and even also in a domain of the complex $\cos\theta$ plane surrounding the physical region.

C. They simplify the unitarity condition. (In fact, since angular momentum is a constant of the motion, they reduce by two the dimensionality of the integrals occurring in the unitarity condition.)

The property A enables one to fix attention only on a small number of partial waves: One can find reasons for supposing these to be much larger than others. A corollary is that the partial-wave series is almost certain to converge in the physical region of $\cos\theta$, though to extend the domain of convergence into the complex plane, as in B, one needs information about the analyticity properties of the complete amplitude (the Lehmann ellipse); there must be no branch point in the physical region.

In Sec. 2 of this paper we investigate partial amplitudes for production processes with the requirement that they satisfy the properties A, B, C above. It is found that these requirements lead to an essentially unique definition of the amplitudes; one arrives, in fact, at the prescription described in some generality by Gunson and Taylor.[3,4] A concrete definition is given in Sec. 3 and in Sec. 4 an application is made as an example. It is shown that the two-particle branch points of the complete amplitude are two sheeted. This is well known for scattering amplitudes,[5] which immediately leads one to suspect that it is true also for production amplitudes. (Certainly in perturbation theory the nature of branch points is independent of the number of external particles.) For simplicity, the exposition of Secs. 2, 3, and 4 is confined to the case of spinless particles and of three particles in the final state. In Sec. 5 we discuss the extension to the case when particles with spin are involved; this involves projecting the spins of the particles on to a "moving" axis. In Sec. 6 we briefly mention some complications provided by extra particles in the final state.

We have said that we shall, in particular, require convergence of the partial-wave series in some domain surrounding the physical region. In some applications one may be content with convergence for real, physical values of the variables. After all, these are the only values that have anything to do with physics and the procedure of analytic continuation to the complex plane is only a mathematical tool. We shall not enter into a discussion of this here because the necessary mathematical conditions for the convergence of a partial-wave series, even in the physical region, seem to be rather uncertain.[6] Even when there is the limited

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[1] For a recent review see "Lectures on Strong Interaction Theory," given by G. F. Chew, Cambridge, 1962 (to be published).
[2] P. V. Landshoff and S. B. Treiman, Nuovo Cimento **19**, 1249 (1961).
[3] J. Gunson and J. G. Taylor, Phys. Rev. **119**, 1121 (1960); J. Gunson (unpublished).
[4] While this paper was being written a short discussion appeared by J. Werle, Phys. Letters **4**, 127 (1963).
[5] W. Zimmermann, Nuovo Cimento **21**, 249 (1961).
[6] *Sufficient* conditions are given, for example, by R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1953), Vol. I, p. 513.

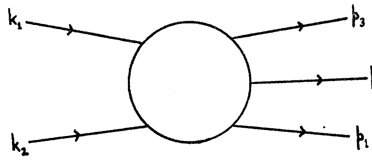


FIG. 1. The production process under discussion.

convergence there remains the question of whether one is permitted to perform the operations (multiplication of partial-wave series and interchange of order of integration and summation) that lead to property C.

We have said that the analytic properties of production amplitudes are complicated. Analytic functions have many powerful properties, such as being determined completely by specification of their values in a very small region, but in practice the only useful expression of analyticity properties seems to be a dispersion relation. It is desirable that such a dispersion relation involve only integrals over real contours and this is not easy to achieve for production amplitudes.[2] In Sec. 7, however, we give a prescription which, at least in perturbation theory, does yield such a dispersion relation for the complete amplitude and would seem to be useful for computational purposes. The partial amplitudes probably satisfy similar dispersion relations and in Sec. 8 we describe a method for determining the positions of the singularities of the partial amplitudes.

2. KINEMATICS AND DYNAMICS

In this and the following two sections we confine ourselves to a discussion of the case in which two spinless particles come in and three emerge (Fig. 1). The momenta are as labeled in the figure and we take

$$p_i^2=m_i^2, \quad i=1, 2, 3.$$

There are five independent scalar variables which may be chosen in a large number of ways. We shall define the complete energy

$$s=(k_1+k_2)^2 \tag{2.1}$$

and the three partial energies

$$s_i=(p_j+p_k)^2. \tag{2.2}$$

These satisfy a linear relation

$$s=s_1+s_2+s_3=(m_1^2+m_2^2+m_3^2)\,, \tag{2.3}$$

so that two more independent variables would be needed to form a complete set.

To define a partial amplitude one integrates out some of the variables, using suitable weight functions, and so is left with a function of ν continuous variables $(1\leqslant\nu\leqslant 5)$ labeled by $(5-\nu)$ discrete indices. A common choice is to take as the continuous variables s and s_1, leaving three discrete indices. One of the latter will certainly be the total angular momentum J, while the other two will be related to the spin of the (p_2p_3) pair and the orbital angular momentum of this pair with respect to p_1. A complication is that the spin of the (p_2p_3) pair is naturally measured in the center-of-mass system of the particles p_2, p_3, while the orbital angular momentum is naturally measured in the over-all center-of-mass, and it is not simple to combine angular momenta in different Lorentz frames.[7] Thus, instead of the spin of the (p_2p_3) pair, it is actually convenient to use its helicity.[8]

This prescription is very useful for an approximate calculation when the partial energy s_1 is chosen to be near a resonance of the (p_2p_3) system and all other interactions in the final state can be neglected. Otherwise, the final-state interactions lead to difficulties. Consider the singularities represented in Fig. 2, each of which corresponds to final-state scattering and represents a singularity dependent on the variable s_3. There will be similar diagrams for singularities depending on s_2.

Figure 2 (a) gives a singularity at

$$s_3=(m_1+m_2)^2.$$

For fixed s,s_1 this does not represent a physical value of s_3, except that when

$$m_2s-(m_1+m_2)s_1-m_1(m_1m_2+m_2^2-m_3^2)=0, \tag{2.4}$$

it is on the edge of the physical region. Thus, (2.4) represents a surface of "end-point" singularity of the partial amplitude, since the latter is defined by an integral of the complete amplitude over the physical region. Beyond the unfamiliar situation of leading to physical region singularities of the partial amplitude, the diagram of Fig. 2(a) does not lead to much trouble—provided we can find the discontinuity across the corresponding cut. It does give difficulties, however, if the

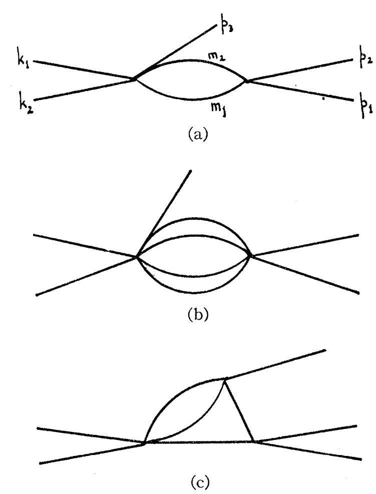


FIG. 2. Some of the singularities that may occur in the physical region.

[7] A. Macfarlane, Rev. Mod. Phys. **34**, 41 (1962).
[8] G. C. Wick, Ann. Phys. (N. Y.) **18**, 65 (1962); L. Cook and B. Lee, Phys. Rev. **127**, 283 (1962).

two internal particles m_1,m_2 can be replaced by two other particles, of total mass M greater than (m_1+m_2). The situation is similar for a multiparticle intermediate state, Fig. 2(b), where we again call $M>(m_1+m_2)$ the total mass in the intermediate state. For sufficiently large s, the corresponding singularity $s_3=M^2$ comes *inside* the physical region. Thus, the condition B of Sec. 1 clearly cannot be met. Similar troubles come from "anomalous" singularities in the physical region,[9] for example, the diagram of Fig. 2(c). As yet, little is known as to how widespread is the occurrence of these.

The situation, then, is that it *may* be possible to find some range of the variables s, s_1 such that the requirements of Sec. 1 are met, but even this is not certain. What is sure is that the range is not very extensive. To get over the difficulties one retains all the partial energies s_i as continuous variables, with two discrete indices. When the s_i are specified, the three-momenta in the over-all center-of-mass system are fixed, so that the final state may be regarded as a rigid body. To arrive at the partial amplitude from the complete amplitude, one integrates over two variables, which may conveniently be angles describing the orientation of the initial momentum with respect to the rigid body. Except for kinematical singularities resulting from an unfortunate choice of these angles, one then has analyticity in a domain in the four-dimensional complex space of these two variables inside which the physical region is contained.[10]

3. PARTIAL-WAVE AMPLITUDES

In this section we describe the "rigid body" partial-wave amplitudes for three particles in the final state. We give the consequences of parity conservation and identity of particles, but for clarity of exposition defer the question of spin to a later section. The work is, in fact, a more detailed but less general exposition of something first done by Gunson and Taylor.[3,4] We imitate many of the techniques of Jacob and Wick,[11] and follow closely their notation and also that in Rose's book on angular momentum.[12]

We first quote our main result. Take any polar axis and azimuth plane fixed with respect to the final momenta in the over-all center-of-mass system, and define corresponding polar and azimuth angles Θ, Φ for the initial momentum. Our partial-wave expansion for the amplitude is

$$B(s,s_i;\,\Theta\Phi)=\sum_{\substack{J\\|\Lambda|\leqslant J}} B_{J\Lambda}(s,s_i)Y_{J\Lambda}^*(\Theta,\Phi). \quad (3.1)$$

The main object of this section is to show that J refers to the total angular momentum and Λ to its component in the direction of the polar axis. The inverse formula is

$$B_{J\Lambda}(s,s_i)=\int d(\cos\Theta)d\Phi\, B(s,s_i;\,\Theta\Phi)Y_{J\Lambda}(\Theta,\Phi). \quad (3.2)$$

The effects of parity conservation and of identity of particles depend upon the choice of polar axis and are considered at the end of this section.

First, consider the definition and normalization of the three-particles states. The normalization convention to be used is contained in the completeness relation

$$\int |\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\rangle\frac{d^3p_1}{2\omega_1}\frac{d^3p_2}{2\omega_2}\frac{d^3p_3}{2\omega_3}\langle\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|=I(3). \quad (3.3)$$

Here $I(3)$ is the projection operator on to the three-particle states in question, and the ω_i are the energies of the individual particles.

We shall only require a particular projection of the relation (3.3) as we work in the center-of-mass system with given total energy E. Choose a system of axes $Oxyz$ fixed in space and so define a set of Euler angles α, β, γ to specify the orientation of some set of "moving" axes $OXYZ$ fixed with respect to the rigid framework of the three center-of-mass momenta. Then we may label the states by E, ω_1, ω_2, α, β, γ and we have

$$\frac{1}{8}\int |E;\,\omega_1\omega_2;\,\alpha\beta\gamma\rangle d\omega_1 d\omega_2 d\alpha d(\cos\beta)d\gamma\langle E;\,\omega_1\omega_2;\,\alpha\beta\gamma| \\ =\delta^3(\mathfrak{P})\delta(\mathfrak{E}-E)I(3), \quad (3.4)$$

where $(\mathfrak{E},\mathfrak{P})$ are the total energy-momentum operators. Alternatively, we may use the labels s, s_1, s_2, α, β, γ:

$$\frac{1}{32s}\int |s,s_i;\,\alpha\beta\gamma\rangle ds_1 ds_2 d\alpha d(\cos\beta)d\gamma\langle s,s_i;\,\alpha\beta\gamma| \\ =\delta^3(\mathfrak{P})\delta(\mathfrak{E}-\sqrt{s})I(3). \quad (3.5)$$

The angular-momentum states will be defined by

$$|s,s_i;\,J\Lambda M\rangle=\frac{2J+1}{8\pi^2}\int d\alpha d(\cos\beta)d\gamma \\ D_{M\Lambda}{}^{J*}(\alpha\beta\gamma)|s,s_i;\,\alpha\beta\gamma\rangle. \quad (3.6)$$

Following the notation of Rose's book,[12] the rotation matrices are defined by

$$D_{MM'}{}^{J}(\alpha\beta\gamma)=\langle JM|R_{\alpha\beta\gamma}|JM'\rangle, \quad (3.7)$$

where the state vectors are ordinary (normalized) angular-momentum states, and $R_{\alpha\beta\gamma}$ is the rotation operator for a rotation designated by Euler angles α, β, γ. Using (3.7) and the unitary property $R^\dagger R=1$, one can deduce the effect of any rotation on the state defined

[9] P. V. Landshoff, Phys. Letters **3**, 116 (1962).

[10] This result follows from the Jost-Lehmann-Dyson representation. See R. Ascoli, Nuovo Cimento **18**, 754 (1960).

[11] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

[12] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957). Note that our convention is that, if the effect of a rotation through Euler angles α, β, γ applied to a vector $\mathbf{q}$ (rather than to the axes) is to transform it into $\mathbf{q}'$, then $R_{\alpha\beta\gamma}|\mathbf{q}\rangle=|\mathbf{q}'\rangle$.

in (3.6):

$$R_{\xi\eta\zeta}|s,s_i;J\Lambda M\rangle=\sum_{|M'|\leqslant J} D_{M'M}{}^J(\xi\eta\zeta)|s,s_i;J\Lambda M'\rangle. \quad (3.8)$$

This equation tells us that J is the total angular momentum and M its component along the (fixed) Oz axis. Λ may be visualized as the component of angular momentum along the (moving) OZ axis. This may be seen most easily by realizing that the functions $D_{M\Lambda}{}^J$ are familiar as the wave functions for a symmetric top, where the quantum numbers have the same interpretation.

The completeness property of the rotation matrices,

$$\sum_J \sum_{|M|\leqslant J} \sum_{|\Lambda|\leqslant J} (2J+1)D_{M\Lambda}{}^J(\alpha\beta\gamma)D_{M\Lambda}{}^{J*}(\alpha'\beta'\gamma') = 8\pi^2\delta(\alpha-\alpha')\delta(\cos\beta-\cos\beta')\delta(\gamma-\gamma'), \quad (3.9)$$

leads to the inverse of Eq. (3.6),

$$|s,s_i;\alpha\beta\gamma\rangle=\sum_J \sum_{|M|\leqslant J} \sum_{|\Lambda|\leqslant J} D_{M\Lambda}{}^J(\alpha\beta\gamma)|s,s_i;J\Lambda M\rangle. \quad (3.10)$$

At this stage it is necessary to introduce the initial two-particle states. Our conventions are summed up in the completeness equation

$$\frac{1}{4}\int |s;\theta\phi\rangle s^{-1/2}kd(\cos\theta)d\phi\langle s;\theta\phi| = \delta^3(\mathfrak{P})\delta(\mathfrak{E}-\sqrt{s})I(2) \quad (3.11)$$

and the equation

$$|s;\theta\phi\rangle=\sum_{J,M} Y_{JM}{}^*(\theta\phi)|s;JM\rangle. \quad (3.12)$$

Here the polar angles θ, ϕ define the orientation of the center-of-mass momentum $\mathbf{k}$ with respect to the fixed axes $Oxyz$. The normalized spherical harmonics are related to the rotation matrices by

$$D_{M0}{}^J(\phi\theta 0)=[4\pi/(2J+1)]^{1/2}Y_{JM}{}^*(\theta\phi).$$

We may now consider the center-of-mass matrix elements

$$\langle s,s_i;\alpha\beta\gamma|T|s;\theta\phi\rangle, \quad (3.13)$$

where $S=1+iT\delta^4(k_1+k_2-p_1-p_2-p_3)$, and transform to the angular momentum representation with the help of (3.10) and (3.12). Since the total-angular-momentum operators commute with the T operator, we may write

$$\langle s,s_i;J'\Lambda M'|T|s;JM\rangle=\delta_{JJ'}\delta_{MM'}B_{J\Lambda}(s,s_i), \quad (3.14)$$

and then (3.13) becomes

$$\sum_{J,M,\Lambda} D_{M\Lambda}{}^{J*}(\alpha\beta\gamma)Y_{JM}{}^*(\theta\phi)B_{J\Lambda}(s,s_i). \quad (3.15)$$

As a consequence of the definition (3.7) of the rotation matrices, (3.15) is equal to the right-hand side of Eq. (3.1), where $(\theta\phi)\to(\Theta\Phi)$ under the rotation[12] $R_{\alpha\beta\gamma}{}^{-1}$. That is to say, the polar angles Θ, Φ are those defined early in this section; and Eq. (3.1) is recovered if the notation $B(s,s_i;\,\Theta\Phi)$ is used for expression (3.13).

Finally, for completeness, we add that Eqs. (3.5) and (3.11) imply that the differential cross section is

$$\frac{\partial^4\sigma}{\partial s_1\partial s_2\partial(\cos\Theta)\partial\Phi}=\frac{\pi^3}{16s^{3/2}k}|B(s,s_i;\,\Theta\Phi)|^2. \quad (3.16)$$

Particular Coordinate Systems

Thus far, we have not used any particular definition of the axes $OXYZ$ fixed relative to the final momenta. The most convenient choice will depend upon the problem at hand, but we discuss two obvious possibilities. We give the consequences of parity conservation and identity of particles for each of them.

A. Choose the Z axis along one of the final (center-of-mass) momenta, say p_3, and the Y axis in the plane of the three final momenta. Then the polar angle Θ is simply the angle between p_3 and the initial momentum. Hence, this might be an appropriate choice of coordinate system if one particle were receiving special attention. For example,

$$\frac{\partial\sigma}{\partial(\cos\Theta)}=\int\frac{\partial^4\sigma}{\partial s_1\partial s_2\partial(\cos\Theta)\partial\Phi}ds_1ds_2d\Phi$$

would describe the angular distribution of $\mathbf{p}_3$.

If parity is conserved we must have

$$B(s,s_i;\,\Theta\Phi)=\eta'\eta B(s,s_i;\,\Theta,\,\pi-\Phi), \quad (3.17a)$$

where $\eta(\eta')$ is the product of the intrinsic parities of the initial (final) particles. We deduce from (3.17a) that

$$B_{J\Lambda}(s,s_i)=\eta'\eta B_{J,\,-\Lambda}(s,s_i). \quad (3.18a)$$

Hence, according as $\eta'\eta=\pm1$, the expansion (3.1) contains only $\cos\Lambda\Phi$ or $\sin\Lambda\Phi$.

If the particles with momenta p_1 and p_2 are identical, the symmetrized wave function is simply

$$|s;\,s_1s_2;\,J\Lambda M\rangle+|s;\,s_2s_1;\,J\Lambda M\rangle, \quad (3.19a)$$

but it is not so easy to construct symmetrical states for three identical particles in this coordinate system.

B. A more symmetrical choice of the Z axis, for three-particle systems, is perpendicular to the plane of the three final (center-of-mass) momenta. The X axis is conveniently chosen to bisect the angle between, say, p_1 and p_2. We denote the angles and quantum numbers in this system by $\tilde{\Theta}$, $\tilde{\Phi}$, J, $\tilde{\Lambda}$.

If parity is conserved we have

$$B(s,s_i;\,\tilde{\Theta}\tilde{\Phi})=\eta'\eta B(s,\,s_i;\,\pi-\tilde{\Theta},\,\tilde{\Phi}), \quad (3.17b)$$

or

$$[1-\eta'\eta(-1)^{J+\tilde{\Lambda}}]B_{J\tilde{\Lambda}}(s,s_i)=0. \quad (3.18b)$$

If particles p_1 and p_2 are identical, a symmetrical state is

$$|s,s_1s_2;\,\tilde{\theta}\tilde{\phi}\tilde{\psi}\rangle+|s,\,s_2s_1;\,\pi-\tilde{\theta},\,\pi+\tilde{\phi},\,-\tilde{\psi}\rangle,$$

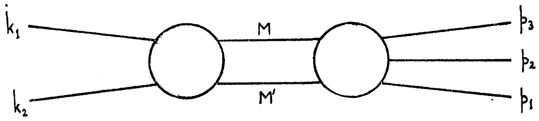


FIG. 3. The two-particle discontinuity, as given by extended unitarity.

or

$$|s,s_1s_2;\,J\tilde{\Lambda}M\rangle+(-1)^J|s,\,s_2s_1;\,J,\,-\tilde{\Lambda},\,M\rangle. \quad (3.19b)$$

Note that the quantum numbers Λ and $\tilde{\Lambda}$ are fundamentally different. The only partial amplitudes that coincide in the two schemes are $J=\Lambda=0$ and $J=\tilde{\Lambda}=0$.

4. AN APPLICATION

We now show, as an application, that the two-particle branch points of production amplitudes are two sheeted, as is the case for scattering.[5] The basic equation for this work is the extended unitarity condition[13] which, in diagrammatical language, gives the discontinuity across a two-particle cut to be Fig. 3. Our method of attack follows closely that of Zimmermann.[5]

As suggested in Fig. 3, we are concerned with the discontinuity across the cut attached to the branch point $s=(M+M')^2$. Extended unitarity gives this discontinuity as[13]

$$B(+)-B(-)=\rho(+)\int A(+)B(-)d\Omega$$
$$=\rho(+)\int A(-)B(+)d\Omega, \quad (4.1)$$

where B, A, respectively, denote the production and scattering amplitudes and

$$\rho(s)=\frac{i\{[s-(M-M')^2][s-(M+M')^2]\}^{1/2}}{8s}.$$

The labels $(+)$ and $(-)$ in (1) indicate whether the functions are to be evaluated with s on top of or underneath the cut. The variables in B other than s are *not* to be taken round cuts [here we have in mind particularly the normal threshold cuts associated with the of partial energies,[2] such as Fig. 2(a)].

We insert in (4.1) the partial-wave decompositions of B and A. For A we make the usual expansion

$$A(s,\omega)=\sum_J (2J+1)A_J(s)P_J(\cos\omega),$$

and for B use Eq. (3.1). Then

$$\begin{aligned}\mathrm{disc}B_{J\Lambda}&=4\pi\rho(+)A_J(+)B_{J\Lambda}(-)=4\pi\rho(+)A_J(-)B_{J\Lambda}(+)\\&=2\pi\rho(+)[A_J(+)B_{J\Lambda}(-)+A_J(-)B_{J\Lambda}(+)]. \quad (4.2)\end{aligned}$$

[13] D. I. Olive (unpublished).

Now define

$$K_{J\Lambda}(s,s_i)=\frac{B_{J\Lambda}(s,s_i)}{1+2\pi\rho(s)A_J(s)}, \quad (4.3)$$

so that the corresponding discontinuity of $K_{J\Lambda}$ is

$$\mathrm{disc}K_{J\Lambda}=\frac{B_{J\Lambda}(+)}{1+2\pi\rho(+)A_J(+)}-\frac{B_{J\Lambda}(-)}{1+2\pi\rho(-)A_J(-)}. \quad (4.4)$$

Using the relation $\rho(+)=-\rho(-)$, we can reduce this to an expression whose numerator is zero because of (4.2), so that $K_{J\Lambda}$ actually does not have the two-particle cut. But,

$$B_{J\Lambda}=K_{J\Lambda}\left\{1+A_J\times\frac{2\pi i[(s-(M-M')^2)(s-(M+M')^2)]^{1/2}}{8s}\right\}, \quad (4.5)$$

hence, since the singularity at $s=(M+M')^2$ of A_J is two sheeted, so is that of $B_{J\Lambda}$.

This completes the proof that the complete amplitude B is also two sheeted, except that we have ignored one question. In order to pass from Eq. (4.1) to (4.2) above we must examine the convergence of the partial-wave expansion of B both at $(+)$ and at $(-)$. This we now do for a particular case. Let P be the production threshold $s=(m_1+m_2+m_3)^2$, which marks the beginning of the physical region for the amplitude B. Suppose that, as in Fig. 4, our branch point $s=(M+M')^2$ is the only singularity on the right-hand cut in the s channel that lies below P. We shall choose to evaluate the discontinuity in (1) an infinitesimal distance below P, as indicated by the positions of $(+)$ and $(-)$ in Fig. 4, with the partial energies fixed an infinitesimal distance below *their* physical thresholds. If we further suppose that these physical thresholds $s_i=(m_j+m_k)^2$ represent the lowest singularities in the s_i channels, time-reversal invariance then requires that B take complex-conjugate values at $(+)$ and $(-)$. Thus, if we prove the convergence of the partial-wave series at $(+)$, it is guaranteed also at $(-)$.

To do this we use the result of Ascoli[10] that for physical values of s,s_i, the complete amplitude B is analytic in some region R of four-dimensional complex $(\cos\Theta,\Phi)$ space analogous to the Lehmann ellipse for scattering. [Ascoli actually works with variables different from $\cos\Theta$, Φ, so we have to check that we have not introduced any kinematic singularities through our

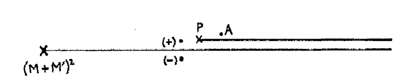


FIG. 4. The branch point $(M+M')^2$ corresponding to Fig. 3 and the physical threshold P, with their attached cuts, drawn in the complex s plane. The discontinuity in Fig. 3 is evaluated between the points $(+)$ and $(-)$.

choice of variables. That we have not may be seen from the fact that there is a one-one correspondence between sets of values of $(s,s_i,\cos\Theta,\Phi)$ and configurations of the momenta.] We start with this result for s at the point A in Fig. 4, an infinitesimal distance above P, and for the s_i just above their thresholds. Then[10] the physical region of $\cos\Theta$, Φ is contained well within R. Since B is an analytic function, the boundary of R can only change infinitesimally if we make an infinitesimal continuation in s, s_i to bring s to the point $(+)$ and the s_i just below their thresholds. Hence, the "physical" region of $\cos\Theta$, Φ is still clear of singularities, which is sufficient[6] to ensure that the partial-wave expansion is absolutely and uniformly convergent at $(+)$. Thus, we may derive (4.2) from (4.1) and also deduce that the two-sheeted property of the partial amplitude is shared by the complete amplitude.

Finally, note that we have shown that the two-particle branch cut in the total energy s is two-sheeted; crossing implies similar properties for the corresponding branch points in the partial energies s_i.

5. INCLUSION OF SPINS

We now briefly discuss one possible method of introducing spins into the partial-wave analysis. We shall work with coordinate system B of Sec. 3, so that the polar axis OZ is perpendicular to the plane of the three final momenta. We introduce quantum numbers λ_1, λ_2, λ_3 that are the components of the spins of the final particles in the direction OZ; these are analogous to the component Λ of the total angular momentum. To describe the spins of the initial particles we use their helicities μ_1, μ_2.

Our main results in this section are the partial-wave expansion and its inverse, Eqs. (5.6) and (5.7), the completeness condition (5.9) on the three-particle states, and an illustrative polarization calculation, Eq. (5.11).

First, we define the three-particle states. As in Sec. 3, define a set $Oxyz$ of axes fixed in space and a set $OXYZ$ of "moving" axes that are fixed relative to the final momenta, with OZ perpendicular to all three of them. Let α, β, γ be the Euler angles that describe the orientation of $OXYZ$ with respect to $Oxyz$, and $R_{\alpha\beta\gamma}$ the corresponding rotation operator.[12] Suppose that the momenta which transform into $\mathbf{p}_i$ under $R_{\alpha\beta\gamma}$ are $\mathbf{p}_i'$. Then Oz is perpendicular to each of the $\mathbf{p}_i'$ and so the direction Oz remains well defined if we make a pure Lorentz transformation that brings any of the $\mathbf{p}_i'$ to rest. Hence, it is useful to define the single-particle state $|\lambda_i\rangle$ to represent the ith particle at rest with the Oz component of its spin having eigenvalue λ_i. Let $L(\mathbf{q})$ represent the operator for the pure Lorentz transformation which takes a particle from rest to a state with momentum $\mathbf{q}$. We can then define the final states in either of the two equivalent ways

$$|s,s_i;\alpha\beta\gamma,\lambda_1\lambda_2\lambda_3\rangle = R_{\alpha\beta\gamma}\prod_i [L(\mathbf{p}_i')|\lambda_i\rangle] = \prod_i [L(\mathbf{p}_i)R_{\alpha\beta\gamma}|\lambda_i\rangle]. \quad (5.1)$$

Note that (5.1) is not a product of independently defined one-particle states, since the angles α, β, γ depend upon the set of all three final momenta for their definition.

Definition (5.1) has the obvious but essential property that

$$|s,s_i;\alpha\beta\gamma,\lambda_1\lambda_2\lambda_3\rangle = R_{\alpha\beta\gamma}|s,s_i,000,\lambda_1\lambda_2\lambda_3\rangle, \quad (5.2)$$

which enables us to define angular momentum states

$$|s,s_i;J\Lambda\lambda_iM\rangle = \frac{2J+1}{8\pi^2}\int d\alpha\, d(\cos\beta)d\gamma \times D_{M\Lambda}^{J*}(\alpha\beta\gamma)|s,s_i;\alpha\beta\gamma,\lambda_i\rangle \quad (5.3)$$

transforming under rotations in exactly the same way as the states in Eq. (3.8). It follows, once again, that J and M represent the length and Oz component of the *total* angular momentum; and Λ may be pictured as its OZ component.

Equation (5.1) tells us that, with $\lambda = (\lambda_1+\lambda_2+\lambda_3)$,

$$|s,s_i;\alpha,\beta,\gamma-2\pi;\lambda_i\rangle = e^{2i\pi\lambda}|s,s_i;\alpha\beta\gamma;\lambda_i\rangle.$$

Therefore, from the γ integration in (5.3), 2Λ is even or odd according as 2λ is even or odd.

For the initial two-particle states we use the helicity states

$$\langle|s;J\mu_jM\rangle = \left(\frac{2J+1}{4\pi}\right)^{1/2}\int d(\cos\theta)d\phi \times D_{M\mu}^{J*}(\phi\theta 0)|s;\theta\phi,\mu_j\rangle, \quad (5.4)$$

where $j=1, 2$ and $\mu=\mu_1-\mu_2$. This definition is in accord with the phase convention for the helicity state of Wick[8] rather than Jacob and Wick.[11]

Inverting Eqs. (5.3) and (5.4) and writing

$$\langle s,s_i;J'\Lambda\lambda_iM'|T|s;J\mu_jM\rangle = \delta_{JJ'}\delta_{MM'}\langle s_i;\Lambda\lambda_i|T_J(s)|\mu_j\rangle,$$

we obtain

$$\langle s,s_i;\alpha\beta\gamma,\lambda_i|T|s,\theta\phi,\mu_j\rangle = \sum_{JM\Lambda}\left(\frac{2J+1}{4\pi}\right)^{1/2} D_{M\Lambda}^{J*}(\alpha\beta\gamma) \times D_{M\mu}^{J}(\phi\theta 0)\langle s_i;\Lambda\lambda_i|T_J(s)|\mu_j\rangle. \quad (5.5)$$

Using the fundamental properties of the rotation matrices [which follow from the definition Eq. (3.7)] and introducing the symbol B for the amplitude, Eq. (5.5) becomes

$$B_{\lambda_1\lambda_2\lambda_3}{}^{\mu_1\mu_2}(s,s_i;\Phi\Theta\Psi) = \sum_{J,\Lambda}\left(\frac{2J+1}{4\pi}\right)^{1/2} D_{\Lambda\mu}^{J}(\Phi\Theta\Psi)\langle s_i;\Lambda\lambda_i|T_J(s)|\mu_j\rangle. \quad (5.6)$$

The Euler angles Φ, Θ, Ψ are the transform of ϕ, θ, 0 under the rotation[12] $R_{\alpha\beta\gamma}{}^{-1}$. Therefore, Θ, Φ are just the polar and azimuth angles of the initial center-of-mass momentum k with respect to $OXYZ$, as in Sec. 3. The angle Ψ corresponds to a rotation of the initial state about the initial momentum. It is not measurable, and occurs merely in a phase factor $e^{-i\mu\Psi}$ throughout the expansion (5.6). It will be relevant only if polarized initial states are being considered, and it must disappear from all final results of calculations of experimental quantities.

The inverse of Eq. (5.6) is

$$\langle s_i;\Lambda\lambda_i|T_J(s)|\mu_j\rangle =\left(\frac{2J+1}{4\pi}\right)^{1/2}\int D_{\Lambda\mu}{}^{J*}(\Phi\Theta\Psi)B_{\lambda_i}{}^{\mu_j}(s,s_i;\Phi\Theta\Psi)\times d(\cos\Theta)d\Phi. \quad (5.7)$$

Parity conservation results in the following generalization of Eq. (3.18b):

$$\langle s,s_i;\Lambda\lambda_i|T_J(s)|\mu_j\rangle =\eta\eta'(-1)^{J+\Lambda}\langle s,s_i;\Lambda\lambda_i|T_J(s)|-\mu_j\rangle. \quad (5.8)$$

The completeness equation for the three-particle angular momentum states follows from the obvious generalization of Eq. (3.5) with the help of Eq. (3.9). It is

$$\sum_{J,\Lambda,M,\lambda_i}\left(\frac{8\pi^2}{2J+1}\right)\int\frac{ds_1ds_2}{32s}|s,s_i;J\Lambda\lambda_iM\rangle\langle s,s_i;J\Lambda\lambda_iM| =\delta^3(\mathfrak{P})\delta(\mathfrak{E}-\sqrt{s})I(3). \quad (5.9)$$

This equation would enable one to express three-particle contributions to the unitarity equation.

The quantum numbers λ_i have an obvious relation to polarization of the final particles perpendicular to their plane in the center-of-mass system. However, polarization in this direction would be rather difficult to measure in an experiment. Therefore, as an illustration of the use of the λ_i, we calculate, in terms of the partial amplitudes, the polarization of one particle, say, particle 3, in the plane perpendicular to its momentum $\mathbf{p}_3$ and the initial momentum $\mathbf{k}$.

It will be convenient to choose the OX axis to be parallel to $\mathbf{p}_3$, instead of the possibility mentioned for coordinate system B in Sec. 3. Then the direction in which we are seeking the polarization, $\mathbf{p}_3\times\mathbf{k}$, has direction ratios (0, $-\cos\Theta$, $\sin\Theta\sin\Phi$) referred to $OXYZ$. The polarization is the expectation value of the spin operator in this direction in the rest frame of particle 3; this direction is well defined, since it is perpendicular to $\mathbf{p}_3$ and so unaffected by the pure Lorentz transformation that carries the center-of-mass frame into this rest frame. Let $B(\lambda_3)$ be the scattering amplitude with only its λ_3 dependence made explicit. For simplicity, we suppose particle 3 to have spin $\frac{1}{2}$. Then the required expectation value is, referring to Eq. (5.1),

$$\begin{aligned}&\tfrac{1}{2}N(\Theta,\Phi)\sum_{\lambda_3,\lambda_3'}B^*(\lambda_3)\langle\lambda_3|R_{\alpha\beta\gamma}{}^{-1}\\&\quad\times[\sin\Theta\sin\Phi\sigma_Z-\cos\Theta\sigma_Y]R_{\alpha\beta\gamma}|\lambda_3'\rangle B(\lambda_3')\\&=\tfrac{1}{2}N(\Theta\Phi)\sum_{\lambda_3,\lambda_3'}B^*(\lambda_3)\langle\lambda_3|\\&\quad\times[\sin\Theta\sin\Phi\sigma_z-\cos\Theta\sigma_y]|\lambda_3'\rangle B(\lambda_3')\\&=\tfrac{1}{2}N(\Theta\Phi)\{[|B(\tfrac{1}{2})|^2-|B(-\tfrac{1}{2})|^2]\sin\Theta\sin\Phi\\&\qquad+2\,\mathrm{Im}[B^*(-\tfrac{1}{2})B(\tfrac{1}{2})]\cos\Theta\},\end{aligned} \quad (5.10)$$

where $N(\Theta,\Phi)=(\cos^2\Theta+\sin^2\Theta\sin^2\Phi)^{1/2}$. Therefore, the required polarization $P(s,s_i;\Theta\Phi)$ is given by

$$\begin{aligned}&\frac{\partial^4\sigma}{\partial s_1\partial s_2\partial(\cos\Theta)\partial\Phi}P(s,s_i;\Theta\Phi)\\&=\frac{\pi^3}{32s^{3/2}kw}N(\Theta\Phi)\sum_{\substack{\mu_1\mu_2\\\lambda_1\lambda_2}}\{\sin\Theta\sin\Phi\\&\quad\times[|B_{\lambda_1\lambda_2,1/2}{}^{\mu_1\mu_2}(s,s_i;\Phi\Theta\Psi)|^2-|B_{\lambda_1\lambda_2,-1/2}{}^{\mu_1\mu_2}|^2]\\&\qquad+2\cos\Theta\,\mathrm{Im}[B_{\lambda_1\lambda_2,-1/2}{}^{\mu_1\mu_2*}B_{\lambda_1\lambda_2,1/2}{}^{\mu_1\mu_2}]\},\end{aligned} \quad (5.11)$$

where w is the total number of initial spin states. Finally, the expansion (5.6) should be inserted into (5.11).

6. EXTRA PARTICLES

We now briefly discuss the question of final states with more than three particles. Coordinate system B is clearly not applicable because the momenta in the final state are no longer in a plane, but system A can be generalized. There is, however, a complication arising from the dimensionality of space. This results in Gram-determinantal relations among the scalar products of the momenta which are quadratic in form. When these are solved square roots appear and so specification of the *values* of the variables does not fix the configuration of the momenta. In effect, one also has to specify the signs in front of the square roots.

This may be illustrated most simply for a four-particle final state. Any specification of the total energy and the five independent partial energies leads to *two* configurations of center-of-mass final momenta; one is the mirror image of the other.

7. DISPERSION RELATIONS

As we have said in the Introduction, a dispersion relation ideally includes only real contour integrals. Another requirement is that for as large a part as possible of the integration contour the integrand be a physical quantity. One's first hope is to fix all the scalar invariants except one and to obtain a dispersion relation in the remaining one. Final-state interactions ("triangle

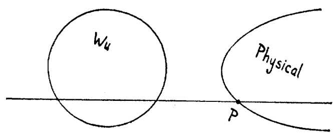


FIG. 5. A schematic diagram of the physical region and the Wu region in the real five-dimensional space of the Lorentz invariants. The straight line defines the variable z in which there is a simple dispersion relation.

singularities") make this impossible, however,[2] because complex singularities inevitably occur.

We here describe a dispersion relation for the complete production amplitude which we hope may prove useful.[14] Its validity, at least in perturbation theory, follows directly from the work of Wu.[15] Let us consider again the simplest production process, Fig. 1. Wu has discovered a certain convex region in the five-dimensional space of the five real independent scalar invariants, which we shall here call the Wu region. The physical region for a given channel stretches away to infinity and is disjoint from the Wu region (see the schematic diagram of Fig. 5). Suppose P is a point on the edge of the physical region, with coordinates

$$s=\sigma,\quad s_i=\sigma_i,\quad t_i=\tau_i,\quad t_i'=\tau_i'. \tag{7.1}$$

Here s, s_i are as defined in Eqs. (2.1) and (2.2) and t_i, t_i' are the momentum transfers

$$t_i=(k_1-p_i)^2,\quad t_i'=(k_2-p_i)^2.$$

Only five of the variables s, s_i, t_i, t_i' are independent, because of linear relations like Eq. (2.3). If we choose fixed real numbers λ_i, μ_i, μ_i' to comply with these relations, then for varying real z, the equations

$$\begin{aligned} s&=\sigma+z,\\ s_i&=\sigma_i+\lambda_i z,\\ t_i&=\tau_i+\mu_i z,\\ t_i'&=\tau_i'+\mu_i' z, \end{aligned} \tag{7.2}$$

define a straight line through P. The relation (2.3), for example, implies

$$\sum_{i=1}^{3}\lambda_i=1.$$

If the fixed numbers are suitably chosen there is a useful dispersion relation in z.

Two points will be considered in making a choice. First, the line will pass through the Wu region, as in Fig. 4. It follows directly from the work of Wu[15] that this is a necessary and sufficient condition for the amplitude, regarded as a function of the single complex variable z, to be a real analytic function–that is, it is real on some part of the real axis in the z plane. Together with the results of Ref. 2, it further follows that this, in turn, is the necessary and sufficient condition for cut-plane analyticity in z, with cuts only on the real axis.

Secondly, for the dispersion relation to be useful one wants the variables to take physical values for $z\geqslant 0$. This demands that

$$\lambda_i>0$$

(or one or more $\lambda_i=0$, but this would make the line miss the Wu region[2]) and that the μ_i, μ_i' have suitable values not greater than zero.

The characterization of the Wu region is not simple[15,16]; in the equal mass case it is bounded by the normal thresholds and triangle singularities in each variable. (In other cases it is more complicated[16]; if the external particles are unstable it will not exist at all.) Since it is convex and symmetrical among the variables, the point

$$s=s_i=t_i=t_i'=\tfrac{3}{2}$$

is at its center (all the masses being equal to unity). Hence, we can easily give an example of a suitable choice of variables:

$$\begin{aligned} s&=9+z,\\ s_i&=4+\tfrac{1}{3}z,\\ t_i&=t_i'=-1-\tfrac{1}{3}z. \end{aligned} \tag{7.3}$$

(This choice corresponds to the three final-state three-momenta being equal and the initial momentum being perpendicular to the production plane.) For this choice of variables some of the simpler singularities in the z plane are shown in Fig. 6. Unfortunately, the discontinuities across the attached cuts, though simple to evaluate,[13] involve integrations of the amplitude for values of the variables not confined to the line defined in (7.3). This will complicate the application of the dispersion relation.

One may expect to find a similar dispersion relation for the partial amplitudes, for which the t_i, t_i' are integrated out. The right-hand cut is presumably again as in Fig. 6, but the left-hand cut will be replaced by the analog of the "circle cut" for scattering partial waves.[17] This cut comes from singularities of the complete amplitude, such as normal thresholds, associated with the variables t_i, t_i'. These produce singularities in the

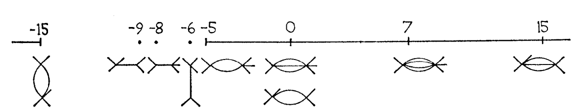


FIG. 6. Some of the singularities in the complex z plane when z is defined by Eqs. (7.3).

[14] Our dispersion relation is different from the parametric dispersion relation of Muraskin and Nishijima [Phys. Rev. **122**, 331 (1961)], which involves integration over values of the invariants that take the momenta off the mass shell.

[15] T. T. Wu, Phys. Rev. **123**, 678 (1961).

[16] J. D. Boyling (unpublished).

[17] S. W. MacDowell, Phys. Rev. **116**, 774 (1960).

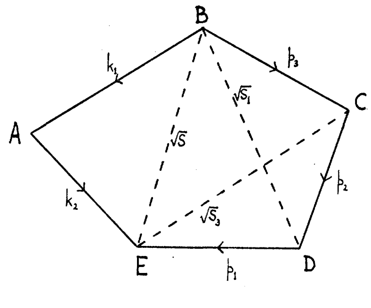


FIG. 7. The momentum vectors for Fig. 1, drawn in four-dimensional complex Euclidean space.

integrals defining the partial amplitudes, by the familiar "end-point" and "pinch" mechanisms. A further discussion of this is the subject of the next section.

8. SINGULARITIES OF THE PARTIAL AMPLITUDES

If one intends to make use of the analytic properties of the partial amplitudes, either by means of a dispersion relation or otherwise, it will be necessary to know their singularities. We here describe a geometrical method for determining their position,[18] though we are not able to supply a general method for deciding on which Riemann sheet they lie, if any. We remark first that some of the singularities of the partial amplitudes [type (b) below] depend on the choice of polar axis. In many applications these can simply be ignored because, although they are important for the individual partial amplitudes, one often performs a summation over the index Λ and then any dynamical singularities associated with the choice of polar axis must disappear.

Consider the vector diagram for the momenta in the process of Fig. 1. The momentum vectors form the pentagon ABCDE of Fig. 7, which is drawn in four-dimensional Lorentz space. The figure is real for physical momenta; we are also concerned with unphysical momenta, in which case the coordinates of the vertexes may become complex. In fact, it will be convenient to draw the figure in Euclidean space, so that the coordinates are complex even for physical momentum. In the center-of-mass system the vector BE, of length $\sqrt{s}$, is in the time direction, while the initial three-momentum $\mathbf{k}$ would be represented by the perpendicular from A on BE. The partial energies s_1, s_3 are represented by the squares of the lengths shown in the figure, while the momentum transfer $t_{13}=(p_3-k_1)^2$ is the square of the length AC.

Our analysis of the singularities of the partial amplitudes, as defined by the integral (3.2), is based on the now familiar lemma of Hadamard applied to multiple integrals.[19] Let $S=0$ denote the various surfaces in $(\cos\Theta, \Phi)$ space that are the singularities of the integrand, that is, of the complete amplitude. (The singularities of the complete amplitude that do not depend on $\cos\Theta$, Φ are carried straight through into the partial amplitudes.) The necessary, though not sufficient, conditions for a given S to produce a singularity of the integral are of two types:

either (a) "pinch":

$$S=\frac{\partial S}{\partial(\cos\Theta)}=\frac{\partial S}{\partial\Phi}=0,$$

or (b) "end point":

$$S=0,\quad \Theta=0 \quad \text{or} \quad \pi.$$

Notice that when $\Theta=0$ or π the coordinate Φ becomes redundant, so that one does not have to include in (b) the extra condition $\partial S/\partial\Phi=0$. Also, since the integral we are considering is, except for a phase factor, independent of choice of azimuth plane, there are no end-point singularities associated with Φ.

A third type of singularity may be produced by two different surfaces S_1 and S_2 touching one another. The equations for this are

(c) $S_1=S_2=0$,

$$\frac{\partial S_1}{\partial(\cos\Theta)}+\lambda\frac{\partial S_2}{\partial(\cos\Theta)}=\frac{\partial S_1}{\partial\Phi}+\lambda\frac{\partial S_2}{\partial\Phi}=0 \text{ for some } \lambda.$$

Notice that, in perturbation theory at least, there is an additional condition on S_1, S_2 that one must correspond to a Landau diagram[20] that is a contraction of the Landau diagram for the other. This is so that the Feynman parameters for S_1, S_2 at the contact be the same. This may be seen to be necessary by inserting the Feynman representation for the amplitude into the integral and then considering the singularities of the resulting multiple integral over $\cos\Theta$, Φ and the Feynman parameters. This argument is probably valid also outside the context of perturbation theory, since Feynman parameters can still be assigned to the S surfaces.[21]

The Eqs. (a), (b), and (c) above can be solved algebraically, but we give here a geometrical method of solution. This is based on the dual diagram construction[20] for the surfaces S. For the simplest example, however, we do not have to consider dual diagrams. This is the case when S corresponds to either a pole or a normal threshold, with equation

$$t_{13}=N^2.$$

Let us examine the conditions (a) above. The equation $S=0$ requires the length of AC in Fig. 7 to be equal to

[18] This method was developed some time ago by one of us in company with J. C. Polkinghorne. We are most grateful to Dr. Polkinghorne for allowing us to include a description here.

[19] J. C. Polkinghorne and G. R. Screaton, Nuovo Cimento **15**, 289 (1960); P. V. Landshoff, J. C. Polkinghorne, and J. C. Taylor, *ibid.* **19**, 939 (1961). In particular, see the Appendix of the paper by J. C. Polkinghorne quoted in Ref. 21.

[20] L. D. Landau, Nucl. Phys. **13**, 181 (1959); J. C. Taylor, Phys. Rev. **117**, 261 (1960).

[21] J. C. Polkinghorne, Nuovo Cimento **23**, 360 (1962).

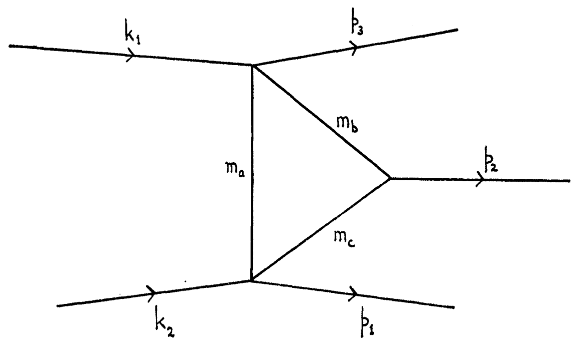


FIG. 8. A triangle singularity.

N, while the other two equations require that it remain equal to N when infinitesimal changes are made in either $\cos\Theta$ or Φ or both. Such changes correspond to a deformation of the pentagon ABCDE subject to the lengths BE, BD, EC remaining fixed. They may be achieved by fixing the points BCDE and displacing A (there are two independent displacements, either in or out of the three-dimensional space defined by BCDE), which amounts to rotating the plane ABE about BE. Such an infinitesimal rotation can only leave AC fixed in length to first order if C lies in the plane ABE. Hence, we must construct ABCE plane and $AC=N$ which, in the equal-mass case, leads to the relation

$$N^2s(3m^2+s_3-s-N^2)=(m^2-s_3)^2m^2. \qquad (8.1)$$

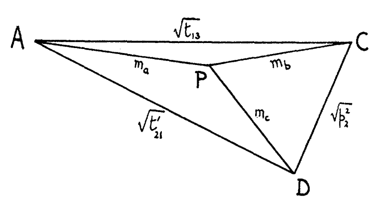
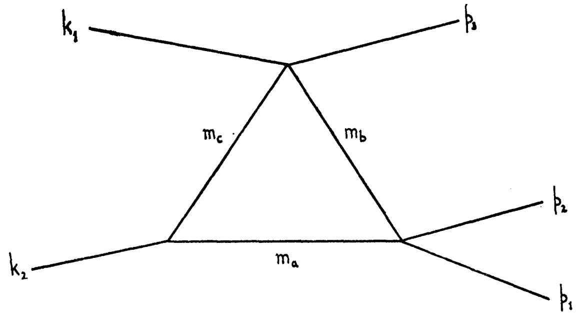


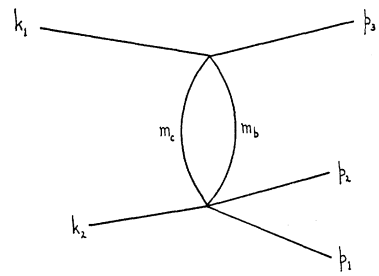
FIG. 9. The dual diagram for Fig. 8.

This equation gives the position of the singularity of the partial amplitude. As we have said, we have no general method for determining the Riemann sheet properties of the singularity.

Although, as is explained above, the type (b) singularity is usually not required, it may also be found. One takes $AC=N$ and draws the momenta in the configuration $\Theta=0$ or π. Suppose, for instance, the polar axis is chosen along $\mathbf{p}_1$ so that $\mathbf{p}_1$ is parallel to the initial



(a)



(b)

FIG. 11. (a) A triangle singularity (b) A normal threshold singularity, obtained by contracting the line m_a in (a).

momentum $\mathbf{k}$ when $\Theta=0$ or π; this corresponds to D being in the plane ABE in Fig. 7. Hence, one obtains a relation among s, s_1, s_3 that is the equation for the singularity of the partial amplitude.

Now we discuss the type (a) singularity corresponding to Fig. 8. This figure represents a singularity of the complete amplitude whose position depends on $t_{13}=(k_1-p_3)^2$ and $t_{21}'=(k_2-p_1)^2$. Its equation $S=0$ may be found from the dual diagram[20] in Fig. 9, where the lines m_a, m_b, m_c represent the internal masses and the figure is drawn in a plane. We now embed Fig. 9 in Fig. 7, as in Fig. 10, and seek the condition that infinitesimal displacements of A with BCDE fixed are compatible with the presence of PA, PD, PC.

Describe the points in the diagram of Fig. 10 by four-vectors with respect to some origin. We make

$$\mathbf{A}\rightarrow\mathbf{A}+\delta\mathbf{A}, \quad \text{so that} \quad \mathbf{P}\rightarrow\mathbf{P}+\delta\mathbf{P}$$

and keep **B**, **C**, **D**, **E** fixed. For PC, PD to remain fixed in length to first order,

$$(\mathbf{P}-\mathbf{C})\cdot\delta\mathbf{P}=0=(\mathbf{P}-\mathbf{D})\cdot\delta\mathbf{P}; \qquad (8.2)$$

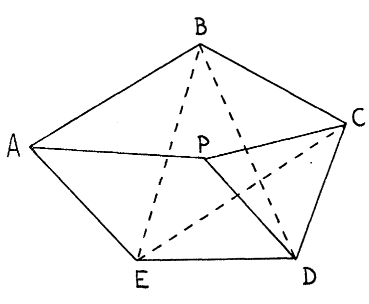


FIG. 10. Fig. 9 embedded in Fig. 7.

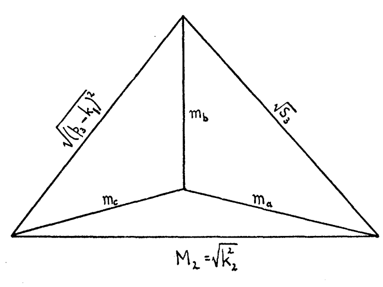


FIG. 12. The dual diagram for Fig. 11(a).

and for PA to be unchanged,

$$(\mathbf{P}-\mathbf{A})\cdot(\delta\mathbf{P}-\delta\mathbf{A})=0. \tag{8.3}$$

But since P, A, C, D are coplanar we may write

$$(\mathbf{P}-\mathbf{A})=\lambda(\mathbf{P}-\mathbf{C})+\mu(\mathbf{P}-\mathbf{D}). \tag{8.4}$$

Insert (8.4) in (8.3) and use (8.2) to get

$$(\mathbf{P}-\mathbf{A})\cdot\delta\mathbf{A}=0. \tag{8.5}$$

Since the possible displacements $\delta\mathbf{A}$ are perpendicular to the plane ABE, the condition (8.5) implies that P lies in this plane. It is now only a matter of geometry to determine what constraint this implies among the lengths BE, BD, CE and so derive the equation involving s, s_1, s_3 corresponding to the desired singularity. This is, of course, a tedious calculation.

Lastly, we give an example of the generation of a type (c) singularity by two different surfaces S_1, S_2. The simplest example is when S_1 is the surface corresponding to Fig. 11(a) and S_2 the surface corresponding to the contraction of Fig. 11(a) drawn in Fig. 11(b). The dual diagram for Fig. 11(a) is drawn in Fig. (12), which is a plane figure. That for Fig. 11(b) is similar, except that the line m_a is omitted and m_b, m_c are collinear. The required value of s_3, representing a singularity of the partial wave, is obtained by including m_a *and* making m_b, m_c collinear. This gives

$$s_3=m_a^2+m_b^2+(m_b/m_c)(m_a^2+m_c^2-M_2^2). \tag{8.6}$$

When (8.6) is satisfied the surface S_1, S_2 actually do more than touch in $(\cos\Theta,\Phi)$ space; they coincide.

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Legendre Transforms and Khuri Representations of Scattering Amplitudes*

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A representation of a scattering amplitude is described in which asymptotic behavior of the Regge type is exhibited in crossing symmetric form. It is based on Legendre transforms, which have similar meromorphy properties to partial wave amplitudes but use variables of the type $(s-2m^2)/2m^2$ instead of the cosine of the scattering angle. The representation obtained is another example of a class that has similar features to the crossing symmetric Sommerfeld-Watson transformation developed by Khuri and based on coefficients of a power series.

A REPRESENTATION that retains the crossing symmetry of the Mandelstam representation while incorporating the high-energy features of scattering amplitudes given by the Regge representation has been recently derived by Khuri.[1] His work helps to provide further justification of an approximation suggested earlier by Chew.[2] It is the purpose of this paper to note that a representation with similar characteristics to that of Khuri can be obtained from Legendre transforms of scattering amplitudes.[3] In particular, a lack of uniqueness is noted and it is suggested that this may give a valuable flexibility for the application of Khuri representations in practical calculations.

Legendre transforms[3] differ from partial wave amplitudes by the choice of variables of integration. For equal masses, define x_i by

$$s=2m^2(1+x_1),\quad t=2m^2(1+x_2),\quad u=2m^2(1+x_3), \tag{1}$$

with $x_1+x_2+x_3=-1$. In the first instance we assume that the Mandelstam representation holds without subtractions when $x_i<1$, $i=1, 2, 3$. Write the amplitude A in three parts corresponding to the three spectral regions, and consider one such part, A_{12} written $A'(x_1,x_2)$,

$$A'(x_1,x_2)=\frac{1}{\pi}\int_1^\infty\frac{dx_1'\,A_1'(x_1',x_2)}{x_1'-x_1}$$
$$=\frac{1}{\pi^2}\int_1^\infty\int_1^\infty\frac{dx_1'dx_2'\rho(x_1',x_2')}{(x_1'-x_1)(x_2'-x_2)}. \tag{2}$$

Define the single Legendre transform $B(l_1,x_2)$ by

$$B(l_1,x_2)=\frac{1}{\pi}\int_1^\infty dx_1'Q_{l_1}(x_1')A_1'(x_1',x_2), \tag{3}$$

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[1] N. Khuri, Phys. Rev. Letters **10**, 420 (1963).
[2] G. F. Chew, Phys. Rev. **129**, 2363 (1963).
[3] R. J. Eden (to be published).